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Multi-shocks in reaction-diffusion models

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Abstract It is shown, concerning equivalent classes, that on a one-dimensional lattice with nearest neighbour interaction, there are only four independent models possessing double-shocks. Evolution of the width of the double-shocks in different models is investigated. Double-shocks may vanish, and the final state is a state with no shock. There is a model for which at large times the average width of double-shocks will become smaller. Although there may exist stationary single-shocks in nearest neighbour reaction diffusion models, it is seen that in none of these models, there exist any stationary double-shocks. Models admitting multi-shocks are classified, and the long period behaviour of multi-shock solutions is also investigated.

PACS. 02.50.Ga Markov processes

1 Introduction

Recently, shocks in one-dimensional reaction-diffusion models have attracted much interest [1–18]. There are some exact results on shocks in one-dimensional reaction-diffusion models as well as simulations, numerical results [6], and mean field results [2]. Formation of localized shocks in one-dimensional driven diffusive systems with spatially homogeneous creation and annihilation of particles is described in [12]. Recently, the families of models with travelling wave solutions on a finite lattice have been presented [4]. These models are the Asymmetric Simple Exclusion Process (ASEP), the Branching Coalescing Random Walk (BCRW), and the Asymmetric Kawasaki-Glauber process (AKGP). In all of these cases the time evolution of the shock observables is equivalent to that of a random walker on a lattice possessing L sites, with homogeneous hopping rates in the bulk, and special reflection rates at the boundary [4]. Shocks have been studied at both the macroscopic and the microscopic levels and there are some efforts on addressing the question as to how macroscopic shocks originate from the microscopic dynamics [7]. Hydrodynamic limits are also investigated.

Among the important aspects of reaction-diffusion systems, is the phase structure of the system. The static phase structure depends on the time-independent profiles of the system, while the dynamical phase structure affects the evolution of the system, especially its relaxation behaviour. In [19–22], the phase structure of some classes of single or multiple species reaction-diffusion systems are considered. These investigations are based on the one-point functions of the systems. In recent work, both stationary and dynamical single-shocks on a one-dimensional lattice are investigated [18]. Both an infinite lattice and a finite lattice with boundaries are considered. Static and dynamical phase transitions of these models have been studied. It is found that the ASEP has no dynamical phase transition, but both the BCRW and the AKGP have three phases, and the system may show dynamical phase transitions [18].

The question addressed in in the present work is, on a one-dimensional lattice with nearest neighbour interaction, which models possess double-shock and multi-shock solutions? Here, double-shock means an uncorrelated state where the occupation probability has two jumps. All the models have nearest neighbour interactions, and are on a one-dimensional lattice. For equivalent classes, it is concluded there are only four independent models possessing double-shocks. For two of the models, double-shock vanishes, and the final state is a linear combination of Bernoulli measures. There is a model for which at large times the average width of double-shock becomes small. It can be easily seen that there may exist stationary single-shocks in nearest-neighbour reaction diffusion models (BCRW, and AKGP), in other words there are single-shock states without any evolution. But in none of these models, there is no stationary double-shock. Combining single-shocks one may construct multi-shocks. There are multi-shocks of the type $(0, \rho, 0, \rho, ...)$ and $(0, 1, 0, 1, ...)$. At large times the final state is a linear combination of single-shocks, or a state with no shock.

2 Notation

Consider a one-dimensional lattice, each point of which is either empty or contains one particle. Let the lattice

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have L sites. An empty state is denoted by $|0\rangle$ and an occupied state is denoted by $|1\rangle$.

$$|0\rangle := \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1\rangle := \begin{pmatrix}0\\1\end{pmatrix}. \tag{1}$$

If the probability that the site i is occupied is ρ_i then the state of that is represented by $\binom{1-\rho_i}{\rho_i}$. The state of the system is characterized by a vector

$$|\mathbb{P}\rangle \in \underbrace{\mathbb{V}\otimes\cdots\otimes\mathbb{V}}_{L}, \tag{2}$$

where $\mathbb{V}$ is a 2-dimensional vector space. All the elements of the vector $|\mathbb{P}\rangle$ are nonnegative, and

$$\langle\mathbb{S}|\mathbb{P}\rangle = 1. \tag{3}$$

Here, $\langle\mathbb{S}|$ is the tensor-product of L covectors $\langle s|$, where $\langle s|$ is a covector the components of which (s_α's) are all equal to one. The evolution of the state of the system is given by

$$|\dot{\mathbb{P}}\rangle = \mathcal{H}\,|\mathbb{P}\rangle, \tag{4}$$

where the Hamiltonian $\mathcal{H}$ is stochastic, by which it is meant that its nondiagonal elements are nonnegative and

$$\langle\mathbb{S}|\,\mathcal{H} = 0. \tag{5}$$

The interaction is nearest-neighbour, if the Hamiltonian is of the form

$$\mathcal{H} = \sum_{i=1}^{L-1} H_{i,i+1}, \tag{6}$$

where

$$H_{i,i+1} := \underbrace{1\otimes\cdots\otimes 1}_{i-1}\otimes H\otimes\underbrace{1\otimes\cdots\otimes 1}_{L-1-i}. \tag{7}$$

Nondiagonal elements of H, shown by ω_{ij}, are reaction rates, hence nonnegative, and its diagonal elements are nonpositive. ω_{ij} is the rate for changes of the configuration of a pair of neighbouring sites from the initial state j to the final state i. The state $|00\rangle$ is labelled as the state 1, $|01\rangle$ as 2, $|10\rangle$ as 3, and finally $|11\rangle$ is the fourth state. Thus, e.g. ω_{23} is the rate for change of configuration $|10\rangle$ to $|01\rangle$, which is the hoping rate to the right.

Any configuration of the system may be represented by the vector $|E_a\rangle$. Hence, the system is spanned by 2^L vectors, $|E_a\rangle(a = 1, 2, ...2^L)$, and any physical state is a linear combination of these vectors:

$$|\mathbb{P}\rangle = \sum_{a=1}^{2^L}\mathcal{P}_a|E_a\rangle, \qquad \text{where} \qquad \sum_{a=1}^{2^L}\mathcal{P}_a = 1. \tag{8}$$

$\mathcal{P}_a$'s are nonnegative real numbers. $\mathcal{P}_a$ is the probability of finding the system in the configuration a.

It is said that the state of the system is a single-shock at the site k if there is a jump in the density at the site k, and the state of the system is represented by a tensor product of the states at each site as

$$|e_k\rangle = u^{\otimes k}\otimes v^{\otimes(L-k)}, \tag{9}$$

where

$$u := \begin{pmatrix}1-\rho_1\\ \rho_1\end{pmatrix} \qquad v := \begin{pmatrix}1-\rho_2\\ \rho_2\end{pmatrix}. \tag{10}$$

It is seen that

$$\langle\mathbb{S}|e_k\rangle = 1. \tag{11}$$

$|e_k\rangle$ represents a state for which the occupation probability for the first k sites is ρ_1, and the occupation probability for the next $L-k$ sites is ρ_2. The set $|e_k\rangle$, $k = 0, 1, ...L$ is not a complete set, but linearly independent.

There are three families of stochastic, one-dimensional, non-equilibrium lattice models, (ASEP, BCRW, AKGP), such that when the initial state of these models is a linear superposition of shock states, the state of the system $|\mathbb{P}\rangle$ remains a linear combination of shock states at the later times. For these models

$$\mathcal{H}|ek\rangle = d|e_{k-1}\rangle + d'|e_{k+1}\rangle - (d+d')|e_k\rangle, \tag{12}$$

where d and d' are some parameters which depend on the reaction rates in the bulk, and the densities ρ_1 and ρ_2. Hence, the span of $|e_k\rangle$'s is an invariant subspace of the Hamiltonian $\mathcal{H}$ of the above mentioned models. It should be noted that the number of $|e_k\rangle$'s is $L+1$, and an arbitrary physical state is not necessarily expressible in terms of $|e_k\rangle$'s.

Next, assume that the initial state of the system is a linear combination of shock states:

$$|\mathbb{P}\rangle(0) = \sum_{k=0}^{L} p_k(0)|e_k\rangle. \tag{13}$$

The p_k's are not necessarily nonnegative; hence, any of them may be greater than one. For such an initial state, the system remains in the sub-space spanned by shock measures.

$$|\mathbb{P}\rangle(t) = \sum_{k=0}^{L} p_k(t)|e_k\rangle. \tag{14}$$

Using (11), it is seen that

$$\sum_{k=0}^{L} p_k(t) = 1. \tag{15}$$

The three models are classified as follows [4]

1. ASEP: the only nonvanishing rates in the bulk are the rates of diffusion to the right ω_{23} and left ω_{32}. In this case the densities can take any value between 0 and 1 ($\rho_1, \rho_2 \neq 0, 1$). d and d' are

$$d = \frac{\rho_1(1-\rho_1)}{\rho_2-\rho_1}(\omega_{23}-\omega_{32})$$
$$d' = \frac{\rho_2(1-\rho_2)}{\rho_2-\rho_1}(\omega_{23}-\omega_{32}). \tag{16}$$

It should be noted that the densities ρ_1, and ρ_2 are also related through

$$\frac{\rho_2(1-\rho_1)}{\rho_1(1-\rho_2)} = \frac{\omega_{23}}{\omega_{32}}. \tag{17}$$

Therefore,

$$d = \frac{\rho_1}{\rho_2}\omega_{23}, \qquad d' = \frac{\rho_2}{\rho_1}\omega_{32}. \tag{18}$$

2. BCRW: the nonvanishing rates are, coalescence (ω_{34}, and ω_{24}), branching (ω_{42} and ω_{43}), and diffusion to the left and right (ω_{32} and ω_{23}). The density ρ_1 can take any value between 0 and 1, but ρ_2 should be zero. These parameters are related through

$$\frac{\omega_{23}}{\omega_{43}} = \frac{\omega_{24}+\omega_{34}}{\omega_{42}+\omega_{43}} = \frac{1-\rho_1}{\rho_1}. \tag{19}$$

The parameters d and d' are

$$d = (1-\rho_1)\omega_{32} + \rho_1\omega_{34}, \\ d' = \frac{\omega_{43}}{\rho_1}. \tag{20}$$

If $\omega_{32} = \omega_{34} = \omega_{43} = \omega_{23} = 0$, and $\omega_{24}/\omega_{42} = (1-\rho)/\rho$, then $d = d' = 0$, and the model allows stationary single-shocks.

3. AKGP: the nonvanishing rates are destruction (ω_{12} and ω_{13}), branching to the left and right (ω_{42} and ω_{43}), and diffusion to the left ω_{32}. The probability densities should always take the values $\rho_1 = 1$ and $\rho_2 = 0$. The hoping parameters are $d = \omega_{13}$, $d' = \omega_{43}$.

3 Double-shocks

The state of a double-shock may be defined through

$$|e_{m,k}\rangle = u^{\otimes m} \otimes v^{\otimes k} \otimes w^{\otimes(L-k-m)}, \qquad m+k \le L, \tag{21}$$

where

$$u := \begin{pmatrix} 1-\rho_1 \\ \rho_1 \end{pmatrix} \qquad v := \begin{pmatrix} 1-\rho_2 \\ \rho_2 \end{pmatrix} \qquad w := \begin{pmatrix} 1-\rho_3 \\ \rho_3 \end{pmatrix}. \tag{22}$$

$|e_{m,k}\rangle$ represents a state for which the occupation probability for the first m sites is ρ_1, the occupation probability for the next k sites is ρ_2, and the occupation probability for remaining sites is ρ_3. A state of this type is a double-shock. It has the first shock at the site m, and the other at the site $m+k$. The width of double-shock is k, and the densities are $\rho_i \in [0,1]$. For it to be classed as a double-shock, ρ_1 should be different from ρ_2, and ρ_2 also should be different from ρ_3. Suitable Hamiltonians for the system possess the property that the span of $|e_{m,k}\rangle$'s is an invariant subspace of $\mathcal{H}$:

$$\mathcal{H}|e_{m,k}\rangle = d_1|e_{m-1,k+1}\rangle + d_1'|e_{m+1,k-1}\rangle + d_2|e_{m,k-1}\rangle \\ + d_2'|e_{m,k+1}\rangle - (d_1+d_1'+d_2+d_2')|e_{m,k}\rangle, \qquad k \ge 2, \tag{23}$$

where the d_i's and d_i''s are parameters which depend on the reaction rates, and may be considered as the rates of jump for the shock to the left or right, respectively. $\mathcal{H}|e_{m,1}\rangle$ will be considered later.

Similar to single-shocks, one should examine cases with different values of ρ separately. The region of values may be divided for ρ into $\rho = 0$, $0 < \rho < 1$, and $\rho = 1$. From now on the cases $\rho = 0$ and $\rho = 1$ will be explicitly stated, and wherever ρ appears, it means that $\rho \neq 0, 1$. Double-shocks may possess various combinations of densities. There are different models, which may transform to each other through particle-hole, or right-left interchange. These models are called equivalent models. As an example, the model allowing the double-shock $(\rho_1, \rho_2, \rho_3) = (0, \rho, 1)$ is related to the model allowing the double-shock $(1, \rho, 0)$ through right-left interchange. It is also related to the model allowing the double-shock $(1, 1-\rho, 0)$ through particle-hole interchange.

Consider the double-shock $(0, 1, \rho)$. A necessary condition for the Hamiltonian where the span of double-shock measures is an invariant subspace of $\mathcal{H}$, is that the span of each of single-shock measures $(0, 1)$ and $(1, \rho)$ are separately invariant subspaces of $\mathcal{H}$. The single-shocks $(0, 1)$ form an invariant subspace for the Hamiltonian in the AKGP. The only interactions which may have nonzero rates are

$$\emptyset A \to (\emptyset\emptyset, AA), \qquad A\emptyset \to (\emptyset\emptyset, AA, \emptyset A). \tag{24}$$

So long as the single-shock $(0, 1)$ is considered, there is no extra constraint on the nonzero reaction rates. The single-shocks $(1, \rho)$ form an invariant subspace for the Hamiltonian in the BCRW, with the following interactions:

$$\emptyset\emptyset \to (\emptyset A, A\emptyset), \qquad \emptyset A \to (\emptyset\emptyset, A\emptyset), \qquad A\emptyset \to (\emptyset\emptyset, \emptyset A), \tag{25}$$

whose reaction rates should satisfy

$$\frac{\omega_{21}+\omega_{31}}{\omega_{12}+\omega_{13}} = \frac{\omega_{23}}{\omega_{13}} = \frac{\rho}{1-\rho}. \tag{26}$$

The space of parameters of this model [a double-shock $(0, 1, \rho)$] is the overlap of the space of parameters of the AKGP and the BCRW. Gathering all these together, it is apparent that all the reaction rates should be zero. Hence, there is no reaction diffusion model with nearest neighbour interaction for which the double-shocks $(0, 1, \rho)$ form an invariant subspace.

It can easily be shown that, concerning equivalent classes, there are only four independent cases.

1. (ρ_1, ρ_2, ρ_3): Among the valid models, (and for $\rho_i = 0, 1$), the ASEP is the only model for which double-shocks forms an invariant subspace. The only nonvanishing rates are ω_{23} and ω_{32}, and they should satisfy

$$\frac{\omega_{23}}{\omega_{32}} = \frac{\rho_2(1-\rho_1)}{\rho_1(1-\rho_2)} = \frac{\rho_3(1-\rho_2)}{\rho_2(1-\rho_3)}. \tag{27}$$

The d_i's and d_i''s are

$$d_1\frac{\rho_2}{\rho_1} = d_2\frac{\rho_3}{\rho_2} = \omega_{23} \\ d_1'\frac{\rho_1}{\rho_2} = d_2'\frac{\rho_2}{\rho_3} = \omega_{32}. \tag{28}$$

This model is investigated by [18]. The rates ω_{23} and ω_{32} are positive and nonzero; hence, the d_i's and d_i''s are also nonzero. For stationary double-shock to occur, $\mathcal{H}|e_{m,k}\rangle = 0$, which makes $d_i = d_i' = 0$. This is unacceptable. Therefore, it is impossible to have stationary double-shock in the ASEP.

2. $(0,\rho,0)$, (or $(\rho,0,\rho)$): The necessary condition for a model possessing double-shocks $(0,\rho,0)$ (or $(\rho,0,\rho)$) is that this model possesses both single-shocks $(0,\rho)$, and $(\rho,0)$. Nonvanishing rates for such a model are ω_{23}, ω_{24}, ω_{32}, ω_{34}, ω_{42} and ω_{43}. These rates should satisfy
$$\frac{\omega_{24}+\omega_{34}}{\omega_{42}+\omega_{43}} = \frac{\omega_{32}}{\omega_{42}} = \frac{\omega_{23}}{\omega_{43}} = \frac{1-\rho}{\rho}. \tag{29}$$
The d_i's and d_i''s are
$$\begin{aligned} d_1 &= \frac{\omega_{42}}{\rho}, & d_1' &= (1-\rho)\omega_{23} + \rho\omega_{24}, \\ d_2' &= \frac{\omega_{43}}{\rho}, & d_2 &= (1-\rho)\omega_{32} + \rho\omega_{34}. \end{aligned} \tag{30}$$
The Hamiltonian with the above mentioned reaction rates also possesses the double-shock $(\rho,0,\rho)$. The only difference is that the rate of jump to the left (and right) of the first double-shock is the rate of jump to the right (and left) for the second one. For stationary double-shocks to occur, the d_i's should be zero, which requires that all the rates are zero. Therefore, there is no stationary double-shock in this model.
3. $(0,1,0)$: The nonvanishing rates are ω_{13}, ω_{12}, ω_{42}, ω_{43}. This model is an asymmetric generalization of the zero temperature Glauber model. The d_i's and d_i''s are
$$\begin{aligned} d_1 &= \omega_{42}, & d_1' &= \omega_{12}, \\ d_2 &= \omega_{13}, & d_2' &= \omega_{43}. \end{aligned} \tag{31}$$
To have stationary double-shock the d_i's and d_i''s should be zero, which makes all the reaction rates vanish. Therefore, this model does not have any stationary double-shock as well.
4. $(0,\rho,1)$: The only nonvanishing rate is ω_{23}. The d_i's and d_i''s are
$$\begin{aligned} d_1 &= 0, & d_1' &= (1-\rho)\omega_{23}, \\ d_2 &= \rho\omega_{23}, & d_2' &= 0. \end{aligned} \tag{32}$$
This model does not have any stationary double-shock as well.

If the initial state is a linear combination of double-shocks, then
$$|\mathcal{P}\rangle(t) = \sum_{m=-\infty}^{\infty}\sum_{k=1}^{\infty} p_{m,k}(t)|e_{m,k}\rangle, \tag{33}$$
where $p_{m,k}$ is the contribution of the double-shock (mk) to the state of the system. Using (4), (33), and the linear independence of $|e_{m,k}\rangle$'s, the dynamical equation can be obtained for the p_{mk}'s. It is difficult to solve difference equations of this type. The variable $p_{m,k}$ has two indices: m representing the position of the first shock, and k the width of the double-shock. The position of the first shock can be neglected, and only the width of double-shock examined. This yields a difference equation which can be solved more easily.

Next, consider the general case where the span of $|e_{mk}\rangle$ is an invariant subspace of $\mathcal{H}$, then
$$\mathcal{H}|e_{m,k}\rangle = \sum_{m',k'} \mathcal{H}_{mk}^{m'k'}|e_{m',k'}\rangle. \tag{34}$$
If the Hamiltonian has the property that $\sum_{m'}\mathcal{H}_{mk}^{m'k'}$ is independent of m, then a new Hamiltonian $\widetilde{\mathcal{H}}$ can be defined using
$$\widetilde{\mathcal{H}}_k^{k'} =: \sum_{m'} \mathcal{H}_{mk}^{m'k'}. \tag{35}$$
It can easily be shown that $\widetilde{\mathcal{H}}$ is stochastic, meaning that
$$\begin{aligned} \widetilde{\mathcal{H}}_k^{k'} &> 0, \qquad \text{for } k \neq k', \\ \textstyle\sum_{k'}\widetilde{\mathcal{H}}_k^{k'} &= 0. \end{aligned} \tag{36}$$
Then, the position of the first shock m can be ignored, leaving only the contribution from double-shocks with width k to consider. Clearly, it is inevitable that some part of information about the position of the first shock will be lost. Now $|f_k\rangle$ can be defined as the state of a double-shock with width k. Identifying all $|e_{m,k}\rangle$ with the same m to each other in the state (33), another state $|\widetilde{\mathbb{P}}\rangle$ can be defined, where the information about the position of the first shock is ignored:
$$|\widetilde{\mathbb{P}}\rangle(t) = \sum_{k=1}^{\infty} q_k(t)|f_k\rangle. \tag{37}$$
Here, $q_k(t)$ is defined using
$$q_k := \sum_{m=-\infty}^{\infty} p_{mk}. \tag{38}$$
This is the contribution of all double-shocks possessing width k. Then, instead of (23),
$$\widetilde{\mathcal{H}}|f_k\rangle = D|f_{k+1}\rangle + D'|f_{k-1}\rangle - (D+D')|f_k\rangle, \qquad k \geq 2. \tag{39}$$
where
$$D := d_1 + d_2', \qquad D' := d_1' + d_2. \tag{40}$$

3.1 Double-shocks $(0,\rho,0)$ and $(0,1,0)$ on a periodic lattice

Next, consider a lattice with L sites, and with periodic boundary conditions. Then, the only double-shocks which can exist are $(0,1,0)$ and $(0,\rho,0)$. Now add up the contributions of all double-shocks with the same width. The position of double-shocks will again be dispersed. Then one should work with $|f_k\rangle$, which represents the state of a double-shock with the width k. $|f_0\rangle$, and $|f_L\rangle$ are

Bernoulli measures corresponding to an empty lattice and a full lattice, respectively. It can easily be shown that

$$\begin{aligned}
&\widetilde{\mathcal{H}}|f_0\rangle = 0,\\
&\widetilde{\mathcal{H}}|f_k\rangle = D|f_{k+1}\rangle + D'|f_{k-1}\rangle - (D+D')|f_k\rangle, \quad k \neq 0, L,\\
&\widetilde{\mathcal{H}}|f_L\rangle = 0.
\end{aligned} \tag{41}$$

Here, D represents the rate at which the width of the double-shock is increasing, and D' represents the rate its width is decreasing. This model can be mapped to a model with one particle on a lattice with boundaries at $k = 0$, and $k = L$. The particle hops to the right and left with the rates D and D', respectively, and there are traps at the boundaries. The system has only two stationary states, $|f_0\rangle$, and $|f_L\rangle$. Hence, at large times there is no shock, and the final state is a linear combinations of the Bernoulli measures:

$$|\widetilde{\mathbb{P}}\rangle = q_0|f_0\rangle + q_L|f_L\rangle. \tag{42}$$

If the initial state is a linear combination of $|f_k\rangle$'s, then

$$|\widetilde{\mathbb{P}}\rangle(t) = \sum_{k=1}^{L} q_k(t)|f_k\rangle. \tag{43}$$

Using (40) yields

$$\begin{aligned}
\dot{q}_0 &= D'q_1,\\
\dot{q}_1 &= D'q_2 - (D+D')q_1,\\
\dot{q}_k &= D'q_{k+1} + Dq_{k-1} - (D+D')q_k,\\
&\qquad\qquad k \neq 0, 1, L-1, L,\\
\dot{q}_{L-1} &= Dq_{L-2} - (D+D')q_{L-1},\\
\dot{q_L} &= Dq_{L-1}.
\end{aligned} \tag{44}$$

The $q_k(t)$'s in the bulk ($k \neq 0, L$) are

$$\begin{aligned}
q_k(t) = &\frac{2}{L}\left(\frac{D}{D'}\right)^{k/2} \exp\left[-\left(D+D'\right)t\right]\\
&\times \sum_{s=1}^{L-1}\sum_{m=1}^{L-1} q_m(0)\left(\frac{D}{D'}\right)^{m/2} \sin\left(\frac{s\pi m}{L}\right)\sin\left(\frac{s\pi k}{L}\right)\\
&\times \exp\left[2t\sqrt{DD'}\cos\left(\frac{s\pi}{L}\right)\right].
\end{aligned} \tag{45}$$

One may integrate $q_1(t)$ and $q_{L-1}(t)$ to obtain $q_0(t)$ and $q_L(t)$, which are the only terms surviving at large times. There is also another way to obtain the q_0, and q_L at infinitely large times. In fact, there are two constants of motion $\mathcal{I}_1$ and $\mathcal{I}_2$. The first of these, $\mathcal{I}_1$ is related to the conservation of probability:

$$\langle\mathbb{S}|\widetilde{\mathbb{P}}\rangle := 1 \quad \Rightarrow \quad \mathcal{I}_1 \sum_{k=0}^{L} q_k(t) = 1, \tag{46}$$

and

$$\mathcal{I}_2 := \sum_{k=0}^{L} q_k(t)\left(\frac{D'}{D}\right)^k. \tag{47}$$

It should be noted that the system has two stationary states. Therefore, there are two right eigenvectors corresponding to zero eigenvalue for the Hamiltonian $\mathcal{H}$, together with two left eigenvectors corresponding to zero eigenvalue for $\mathcal{H}$. These are

$$\langle\mathbb{S}| = (1 \quad 1 \quad 1 ... 1), \tag{48}$$

and

$$\langle\mathbb{S}'| = \left[1 \quad \frac{D'}{D} \quad \left(\frac{D'}{D}\right)^2 \quad \left(\frac{D'}{D}\right)^3 ... \left(\frac{D'}{D}\right)^L\right]. \tag{49}$$

The second constant of motion can be obtained using $\langle\mathbb{S}|\widetilde{\mathbb{P}}\rangle(t)$. So long as $D \neq D'$, the constants of motion $\mathcal{I}_1$ and $\mathcal{I}_2$ are two independent quantities. For $D = D'$, $\mathcal{I}_1$ and $\mathcal{I}_2$ are the same. However, as the stationary state has twofold degeneracy, there should exist another constant of motion. The second independent constant of motion is $\mathcal{I'}_2 := \sum_{k=0}^{L} kq_k(t) = \langle k\rangle$. Hence, for $D = D'$, the average width of the shock, $\langle k\rangle$, is a constant of motion. This is expected because D and D' are the rates for increasing and decreasing the width of the double-shock, respectively.

For the double-shock $(0, \rho, 0)$, $D'/D = 1 - \rho < 1$. Hence, the constants of motion are $\mathcal{I}_1$ and $\mathcal{I}_2$. The first of these $\mathcal{I}_1$ is the summation of probabilities for finding a double-shock with any width; therefore, it should be equal to one. The second constant of motion also has a physical meaning. The rate for changing any configuration of a pair of neighbouring sites to the state $|\emptyset\emptyset\rangle$ is zero. Therefore, the probability for finding a completely empty lattice does not change with time. $\mathcal{I}_2$ is exactly the probability of finding an empty lattice in the initial state:

$$\mathcal{I}_2 = \sum_{k=0}^{L} q_k(t)(1-\rho)^k = \sum_{k=0}^{L} q_k(0)(1-\rho)^k. \tag{50}$$

Using the constants of motion, for $D \neq D'$, at infinitely large times, gives

$$\begin{aligned}
&q_0 + q_L = 1,\\
&q_0 + \left(\frac{D'}{D}\right)^L q_L = \sum_{k=0}^{L} q_k(0)\left(\frac{D'}{D}\right)^k.
\end{aligned} \tag{51}$$

The solution to these equations is

$$\begin{aligned}
q_0(\infty) &= \frac{\sum_{k=0}^{L} q_k(0)(D'/D)^k - (D'/D)^L}{1-(D'/D)^L},\\
q_L(\infty) &= \frac{1 - \sum_{k=0}^{L} q_k(0)(D'/D)^k}{1-(D'/D)^L}.
\end{aligned} \tag{52}$$

Thus, the contribution of $|f_0\rangle$ and $|f_L\rangle$ in the final state depends on both reaction rates and initial conditions.

The Hamiltonian for the model possessing the double-shock $(0, 1, 0)$, with $D = D'$ is the Hamiltonian for the

Glauber model at zero temperature. This model is investigated by [19, 23, 24]. The average density at each site $\langle n_i\rangle(t)$ at the time t, and all the correlation functions at large times for an infinite lattice have been calculated in [23]. Static and dynamical phase transitions of this model have also been studied in [19]. Here, D is not necessarily equal to D'. For $D' > D$ and large L,

$$\begin{aligned} q_0(\infty) &= 1 - \sum_{k=0}^{L} q_k(0)\,(D'/D)^k - (D'/D)^{k-L}, \\ q_L(\infty) &= 1 - q_0(\infty). \end{aligned} \tag{53}$$

If only double-shocks with finite widths make contributions initially, then in the thermodynamic limit ($L\to\infty$) the system will finally fall into the state f_0. However, if $D' < D$ it can be seen that both stationary states make contributions to the final state.

For the case $D = D'$,

$$\begin{aligned} q_0(\infty) &= 1 - \frac{1}{L}\sum_{k=0}^{L} k q_k(0) = 1 - \frac{1}{L}\langle k\rangle, \\ q_L(\infty) &= \frac{1}{L}\sum_{k=0}^{L} k q_k(0) = \frac{1}{L}\langle k\rangle. \end{aligned} \tag{54}$$

This means that at large times the system is fully occupied or empty. The probability of finding a fully occupied lattice at large times is equal to the ratio of initial average width of the double-shock to the size of the lattice.

3.2 Double-shock $(0, \rho, 1)$

Next, consider the double-shock $(0, \rho, 1)$ on an infinite lattice. The only nonvanishing rate is ω_{23}, which by a suitable redefinition of time, can be set equal to 1. Direct calculation gives

$$\begin{aligned} &\mathcal{H}|e_{m,1}\rangle = 0, \\ &\mathcal{H}|e_{m,k}\rangle = (1-\rho)|e_{m+1,k-1}\rangle + \rho|e_{m,k-1}\rangle - |e_{m,k}\rangle, \\ &\qquad\qquad k \neq 1. \end{aligned} \tag{55}$$

It is apparent that there is no probability for width increase. If there is initially a shock $|e_{m,k}\rangle$, then at later times its width becomes smaller, and after long periods there are only double-shocks with unit width. Starting with a linear combination of the shocks, the dynamical equation for the $p_{m,k}$'s is

$$\begin{aligned} \dot p_{m,1} &= (1-\rho)p_{m-1,2} + \rho p_{m,2}, \\ \dot p_{m,k} &= (1-\rho)p_{m-1,k+1} + \rho p_{m,k+1} - p_{m,k}, \qquad k \neq 1. \end{aligned} \tag{56}$$

Defining q_k through (38) gives

$$\begin{aligned} \dot q_1 &= q_2, \\ \dot q_k &= q_{k+1} - q_k, \qquad k \neq 1. \end{aligned} \tag{57}$$

Clearly, if the state of the system initially is a double-shock, e.g. $|e_{M\ K}\rangle$, then at later times there are only double-shocks at the position of the first shock in the range $M \le m \le M + K - 1$, and possessing width $1 \le k \le M + K - m$. Now assume that the initial state is

$$|\widetilde{\mathbb{P}}\rangle = \sum_{k=0}^{L} q_k(0)|f_k\rangle, \tag{58}$$

where $|f_k\rangle$ is the state of double-shocks with the width k. Then,

$$\begin{aligned} \dot q_0 &= 0, \\ \dot q_1 &= q_2, \\ \dot q_k &= q_{k+1} - q_k, \qquad 2 \le k \le L-1, \\ \dot q_L &= -q_L, \\ \dot q_k &= 0, \qquad L+1 \le k. \end{aligned} \tag{59}$$

The above equations show that at large times there are only contributions from the double-shocks with unit width. This set of equations can be solved, yielding the result

$$q_k(t) = \sum_{n=0}^{L-k} q_{k+n}(0)\frac{t^n}{n!}\mathrm{e}^{-t}, \qquad 2 \le k \le L. \tag{60}$$

This, together with $\dot q_1 = q_2$, can be used to obtain $q_1(t)$:

$$\begin{aligned} q_1(t) &= q_1(0) + \sum_{n=0}^{L-2} q_{n+2}(0)\int_0^t \frac{t'^n}{n!}\mathrm{e}^{-t'} \\ &= \sum_{n=1}^{L} q_n(0) - \sum_{n=0}^{L-2}\sum_{m=0}^{n} q_{n+2}(0)\frac{t^m}{m!}\mathrm{e}^{-t}. \end{aligned} \tag{61}$$

As expected, at large times all the double-shocks change into the double-shock with unit width.

Next, examine the distribution of these double-shocks at large times. Using (55), and defining $A_{m,k} := \exp(t\mathcal{H})|e_{mk}\rangle$, it is seen that

$$\begin{aligned} &\frac{\partial A_{m,k}}{\partial t} + A_{m,k} = (1-\rho)A_{m+1,k-1} + \rho A_{m,k-1}, \qquad k \neq 1, \\ &\qquad A_{m,1} = |e_{k1}\rangle. \end{aligned} \tag{62}$$

At large times this equation becomes

$$\begin{aligned} A_{m,k}(\infty) = (1-\rho)A_{m+1,k-1}(\infty) + \rho A_{m,k-1}(\infty), \\ k \neq 1. \end{aligned} \tag{63}$$

The solution is

$$\begin{aligned} A_{m,k}(\infty) &= \lim_{t\to\infty}\left(\mathrm{e}^{t\mathcal{H}}|e_{mk}\rangle\right) \\ &= \sum_{j=1}^{k-1}\binom{k-1}{j}(1-\rho)^j\rho^{k-1-j}|e_{m+j,1}\rangle. \end{aligned} \tag{64}$$

Hence, at large times the state of the system is a linear combination of double-shocks with unit width. The distribution of the position of these double-shocks is a binomial distribution. Now consider the initial state to be a double-shock with width k: $|e_{0,k}\rangle$. Then, the average position of the first shock at large times is

$$\langle j\rangle = (k-1)(1-\rho), \tag{65}$$

and the width of the binomial distribution is $\sqrt{\rho(1-\rho)(k-1)}$.

4 Multi-shocks

Multi-shocks can be constructed by combining single-shocks. The only models with multi-shocks are as follows.

1. $(\rho_1, \rho_2, \rho_3, ...)$: The span of multi-shocks $(\rho_1, \rho_2, \rho_3, ...)$ is an invariant subspace of the ASEP Hamiltonian provided the densities satisfy

$$\frac{\rho_{i+1}(1-\rho_i)}{\rho_i(1-\rho_{i+1})} = \frac{\omega_{23}}{\omega_{32}}. \tag{66}$$

   It can be easily seen that the rate of hoping of the ith shock to the left, d_i, and the rate of hoping of the ith shock to the right, d'_i, is given by

$$d_i = \omega_{23}\frac{\rho_i}{\rho_{i+1}},$$
$$d'_i = \omega_{32}\frac{\rho_{i+1}}{\rho_i}. \tag{67}$$

   This result is obtained in [3].
2. $(0, \rho, 0, \rho, ...)$ and $(0, 1, 0, 1, ...)$: The model allowing multi-shocks of the type $(0, \rho, 0, \rho, ...)$ are ones that allow the double-shocks $(0, \rho, 0)$, or $(\rho, 0, \rho)$. The model possessing multi-shocks of the type $(0, 1, 0, 1, ...)$ is an asymmetric generalization of the zero temperature Glauber model. In both of these multi-shocks there are edges at shock points. The edges are destroyed pairwise. Consider a multi-shock of order N with the first shock at the site m. It is seen that the action of Hamiltonian on such state is

$$\begin{aligned}\mathcal{H}|e_{m,k_1,...,k_{N-1}}\rangle &= d_1|e_{m-1,k_1+1,...,k_{N-1}}\rangle\\ &+ d'_1|e_{m+1,k_1-1,...,k_{N-1}}\rangle\\ &+ d_2|e_{m,k_1-1,...,k_{N-1}}\rangle\\ &+ d'_2|e_{m,k_1+1,...,k_{N-1}}\rangle + ...\\ &+ d_2|e_{m,k_1,...,k_{N-1}-1}\rangle\\ &+ d'_2|e_{m,k_1,...,k_{N-1}+1}\rangle. \end{aligned}\tag{68}$$

   If any of the k_i's on the left hand side is equal to one, then on the right a multi-shock of order $N-2$ will occur. Hence, there is a finite probability that the system transforms into a state with fewer shocks, and there no probability for increasing the number of shocks. In fact, if there exists a state for which any state can transform directly or even indirectly to it, and that state has no evolution, then that state is the final stationary state. Finally, consider a periodic lattice. The number of shocks, N, should be even. Hence at large times the state of system is a state with no shock. For the models on an infinite lattice number of shocks, N may be even or odd. Then for odd N, the final state is a linear combination of single-shocks.

5 Summary

There are three types of models with travelling wave solutions on a one-dimensional lattice. These are classified in [4]. It is seen that there are four types of models allowing double-shocks. Double-shocks, and the models allowing these double-shocks are as follows.

- (ρ_1, ρ_2, ρ_3): the nonvanishing rates are ω_{23}, and ω_{32}.
- $(0, \rho, 0)$, [and also $(\rho, 0, \rho)$]: the nonvanishing rates are ω_{23}, ω_{24}, ω_{32}, ω_{34}, ω_{42} and ω_{43}.
- $(0, 1, 0)$: the nonvanishing rates are ω_{13}, ω_{12}, ω_{42}, ω_{43}.
- $(0, \rho, 1)$: the only nonvanishing rate is is ω_{23}.

There are three type of models allowing multi-shocks. The multi-shocks are of the following types.

- $(\rho_1, \rho_2, \rho_3, ...)$: the nonvanishing rates are ω_{23}, and ω_{32}.
- $(0, \rho, 0, \rho, ...)$: the nonvanishing rates are ω_{23}, ω_{24}, ω_{32}, ω_{34}, ω_{42} and ω_{43}.
- $(0, 1, 0, 1, ...)$: the nonvanishing rates are ω_{13}, ω_{12}, ω_{42}, ω_{43}.

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